A Quantum Dot with Impurity in the Lobachevsky Plane

V. Geyler, P. Šťovíček and M. Tušek

Abstract. The curvature effect on a quantum dot with impurity is investigated. The model is considered on the Lobachevsky plane. The confinement and impurity potentials are chosen so that the model is explicitly solvable. The Green function as well as the Krein Q-function are computed.

Keywords. quantum dot, Lobachevsky plane, point interaction, spectrum.

1. Introduction

Physically, quantum dots are nanostructures with a charge carriers confinement in all space directions. They have an atom-like energy spectrum which can be modified by adjusting geometric parameters of the dots as well as by the presence of an impurity. Thus the study of these dependencies may be of interest from the point of view of the nanoscopic physics.

A detailed analysis of three-dimensional quantum dots with a short-range impurity in the Euclidean space can be found in [1]. Therein, the harmonic oscillator potential was used to introduce the confinement, and the impurity was modeled by a point interaction (δ-potential). The starting point of the analysis was derivation of a formula for the Green function of the unperturbed Hamiltonian (i.e., in the impurity free case), and application of the Krein resolvent formula jointly with the notion of the Krein Q-function.

The current paper is devoted to a similar model in the hyperbolic plane. The nontrivial hyperbolic geometry attracts regularly attention, and its influence on the properties of quantum-mechanical systems has been studied on various models (see, for example, [2, 3, 4]). Here we make use of the same method as in [1] to investigate a quantum dot with impurity in the Lobachevsky plane. We will introduce an appropriate Hamiltonian in a manner quite analogous to that of [1] and derive an explicit formula for the corresponding Green function. In this sense,

our model is solvable, and thus its properties may be of interest also from the mathematical point of view.

During the computations to follow, the spheroidal functions appear naturally. Unfortunately, the notation in the literature concerned with this type of special functions is not yet uniform (see, e.g., [5] and [6]). This is why we supply, for the reader's convenience, a short appendix comprising basic definitions and results related to spheroidal functions which are necessary for our approach.

2. A quantum dot with impurity in the Lobachevsky plane

2.1. The model

Denote by (ϱ, ϕ), $0 < \varrho < \infty$, $0 \leq \phi < 2\pi$, the geodesic polar coordinates on the Lobachevsky plane. Then the metric tensor is diagonal and reads

$$(g_{ij}) = \mathrm{diag}\Big(1, a^2 \sinh^2 \frac{\varrho}{a}\Big)$$

where a, $0 < a < \infty$, denotes the so called curvature radius which is related to the scalar curvature by the formula $R = -2/a^2$. Furthermore, the volume form equals $dV = a \sinh(\varrho/a) \mathrm{d}\varrho \wedge \mathrm{d}\phi$. The Hamiltonian for a free particle of mass $m = 1/2$ takes the form

$$H^0 = -\left(\Delta_{LB} + \frac{1}{4a^2}\right) = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} - \frac{1}{4a^2}$$

where Δ_{LB} is the Laplace-Beltrami operator and $g = \det g_{ij}$. We have set $\hbar = 1$.

The choice of a potential modeling the confinement is ambiguous. We naturally require that the potential takes the standard form of the quantum dot potential in the flat limit ($a \to \infty$). This is to say that, in the limiting case, it becomes the potential of the isotropic harmonic oscillator $V_\infty = \frac{1}{4}\omega^2\varrho^2$. However, this condition clearly does not specify the potential uniquely. Having the freedom of choice let us discuss the following two possibilities:

$$\text{a)} \quad V_a(\varrho) = \tfrac{1}{4}\, a^2\omega^2 \tanh^2 \tfrac{\varrho}{a}, \tag{2.1}$$

$$\text{b)} \quad U_a(\varrho) = \tfrac{1}{4}\, a^2\omega^2 \sinh^2 \tfrac{\varrho}{a}. \tag{2.2}$$

Potential V_a is the same as that proposed in [7] for the classical harmonic oscillator on the Lobachevsky plane. With this choice, it has been demonstrated in [7] that the model is superintegrable, i.e., there exist three functionally independent constants of motion. Let us remark that this potential is bounded, and so it represents a bounded perturbation to the free Hamiltonian. On the other hand, the potential U_a is unbounded. Moreover, as shown below, the stationary Schrödinger equation for this potential leads, after the partial wave decomposition, to the differential equation of spheroidal functions. The current paper concentrates exclusively on case b).

The impurity is modeled by a δ-potential which is introduced with the aid of self-adjoint extensions and is determined by boundary conditions at the base

point. We restrict ourselves to the case when the impurity is located in the center of the dot ($\varrho = 0$). Thus we start from the following symmetric operator:

$$H = -\left(\frac{\partial^2}{\partial\varrho^2} + \frac{1}{a}\coth\!\left(\frac{\varrho}{a}\right)\frac{\partial}{\partial\varrho} + \frac{1}{a^2}\sinh^{-2}\!\left(\frac{\varrho}{a}\right)\frac{\partial^2}{\partial\phi^2} + \frac{1}{4a^2}\right) + \frac{1}{4}\,a^2\omega^2\sinh^2\!\left(\frac{\varrho}{a}\right),$$
$$\mathrm{Dom}(H) = C_0^\infty((0,\infty)\times S^1) \subset L^2\left((0,\infty)\times S^1, a\,\sinh\!\left(\frac{\varrho}{a}\right)\mathrm{d}\varrho\,\mathrm{d}\phi\right). \tag{2.3}$$

2.2. Partial wave decomposition

Substituting $\xi = \cosh(\varrho/a)$ we obtain

$$H = \frac{1}{a^2}\left[(1-\xi^2)\frac{\partial^2}{\partial\xi^2} - 2\xi\frac{\partial}{\partial\xi} + (1-\xi^2)^{-1}\frac{\partial^2}{\partial\phi^2} + \frac{a^4\omega^2}{4}(\xi^2-1) - \frac{1}{4}\right] =: \frac{1}{a^2}\tilde{H},$$
$$\mathrm{Dom}(H) = C_0^\infty((1,\infty)\times S^1) \subset L^2\left((1,\infty)\times S^1, a^2\mathrm{d}\xi\,\mathrm{d}\phi\right). \tag{2.4}$$

Using the rotational symmetry which amounts to a Fourier transform in the variable ϕ, $\tilde{H}$ may be decomposed into a direct sum as follows

$$\tilde{H} = \bigoplus_{m=-\infty}^{\infty} \tilde{H}_m,$$
$$\tilde{H}_m = -\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial}{\partial\xi}\right) + \frac{m^2}{\xi^2-1} + \frac{a^4\omega^2}{4}(\xi^2-1) - \frac{1}{4},$$
$$\mathrm{Dom}(\tilde{H}_m) = C_0^\infty(1,\infty) \subset L^2((1,\infty),\mathrm{d}\xi).$$

Note that $\tilde{H}_m$ is a Sturm-Liouville operator.

Proposition 2.1. *$\tilde{H}_m$ is essentially self-adjoint for $m \neq 0$, $\tilde{H}_0$ has deficiency indices* $(1,1)$.

Proof. The operator $\tilde{H}_m$ is symmetric and semibounded, and so the deficiency indices are equal. If we set

$$\mu = |m|,\ 4\theta = -\frac{a^4\omega^2}{4},\ \lambda = -z - \frac{1}{4},$$

then the eigenvalue equation

$$\left(-\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial}{\partial\xi}\right) + \frac{m^2}{\xi^2-1} + \frac{a^4\omega^2}{4}(\xi^2-1) - \frac{1}{4}\right)\psi = z\psi \tag{2.5}$$

takes the standard form of the differential equation of spheroidal functions (A.1). According to chapter 3.12, Satz 5 in [6], for $\mu = m \in \mathbb{N}_0$ a fundamental system $\{y_{\mathrm{I}},\ y_{\mathrm{II}}\}$ of solutions to equation (2.5) exists such that

$$y_{\mathrm{I}}(\xi) = (1-\xi)^{m/2}\,\mathfrak{P}_1(1-\xi), \quad \mathfrak{P}_1(0) = 1,$$
$$y_{\mathrm{II}}(\xi) = (1-\xi)^{-m/2}\mathfrak{P}_2(1-\xi) + A_m\,y_{\mathrm{I}}(\xi)\log(1-\xi),$$

where, for $|\xi - 1| < 2$, $\mathfrak{P}_1, \mathfrak{P}_2$ are analytic functions in ξ, λ, θ; and A_m is a polynomial in λ and θ of total order m with respect to λ and $\sqrt{\theta}$; $A_0 = -1/2$.

Suppose that $z \in \mathbb{C}\backslash\mathbb{R}$. For $m = 0$, every solutions to (2.5) is square integrable near 1; while for $m \neq 0$, y_{I} is the only one solution, up to a factor, which is square integrable in a neighborhood of 1. On the other hand, by a classical analysis due to Weyl, there exists exactly one linearly independent solution to (2.5) which is square integrable in a neighborhood of ∞, see Theorem XIII.6.14 in [8]. In the case of $m = 0$ this obviously implies that the deficiency indices are $(1, 1)$. If $m \neq 0$ then, by Theorem XIII.2.30 in [8], the operator $\tilde{H}_m$ is essentially self-adjoint. □

Define the maximal operator associated to the formal differential expression

$$L = -\frac{\partial}{\partial\xi}\left((\xi^2 - 1)\frac{\partial}{\partial\xi}\right) + \frac{a^4\omega^2}{4}(\xi^2 - 1) - \frac{1}{4}$$

as follows

$$\begin{aligned}\mathrm{Dom}(H_{max}) = \Bigg\{ & f \in L^2((1,\infty), \mathrm{d}\xi) :\ f, f' \in AC((1,\infty)),\\ & -\frac{\partial}{\partial\xi}\left((\xi^2 - 1)\,\frac{\partial f}{\partial\xi}\right) + \frac{a^4\omega^2}{4}\,(\xi^2 - 1) f \in L^2((1,\infty), \mathrm{d}\xi)\Bigg\},\\ H_{max} f = L f. &\end{aligned}$$

According to Theorem 8.22 in [9], $H_{max} = \tilde{H}_0^\dagger$.

Proposition 2.2. *Let $\kappa \in (-\infty, \infty\,]$. The operator $\tilde{H}_0(\kappa)$ defined by the formulae*

$$\mathrm{Dom}(\tilde{H}_0(\kappa)) = \{f \in \mathrm{Dom}(H_{max}) :\ f_1 = \kappa f_0\}\,,\ \tilde{H}_0(\kappa) f = H_{max} f,$$

where

$$f_0 := -4\pi a^2 \lim_{\xi\to 1+} \frac{f(\xi)}{\log(2a^2(\xi - 1))}\,,\ f_1 := \lim_{\xi\to 1+} f(\xi) + \frac{1}{4\pi a^2}\, f_0 \log\big(2a^2(\xi - 1)\big),$$

is a self-adjoint extension of $\tilde{H}_0$. There are no other self-adjoint extensions of $\tilde{H}_0$.

Proof. The methods to treat δ like potentials are now well established [10]. Here we follow an approach described in [11], and we refer to this source also for the terminology and notations. Near the point $\xi = 1$, each $f \in \mathrm{Dom}(H_{max})$ has the asymptotic behavior

$$f(\xi) = f_0\, F(\xi, 1) + f_1 + o(1) \quad \text{as } \xi \to 1+$$

where $f_0, f_1 \in \mathbb{C}$ and $F(\xi, \xi')$ is the divergent part of the Green function for the Friedrichs extension of $\tilde{H}_0$. By formula (2.11) which is derived below, $F(\xi, 1) = -1/(4\pi a^2)\log\big(2a^2(\xi - 1)\big)$. Proposition 1.37 in [11] states that $(\mathbb{C}, \Gamma_1, \Gamma_2)$, with $\Gamma_1 f = f_0$ and $\Gamma_2 f = f_1$, is a boundary triple for H_{max}.

According to theorem 1.12 in [11], there is a one-to-one correspondence between all self-adjoint linear relations κ in $\mathbb{C}$ and all self-adjoint extensions of $\tilde{H}_0$

given by $\kappa \longleftrightarrow \tilde{H}_0(\kappa)$ where $\tilde{H}_0(\kappa)$ is the restriction of H_{max} to the domain of vectors $f \in \mathrm{Dom}(H_{max})$ satisfying

$$(\Gamma_1 f, \Gamma_2 f) \in \kappa. \tag{2.6}$$

Every self-adjoint relation in $\mathbb{C}$ is of the form $\kappa = \mathbb{C}v \subset \mathbb{C}^2$ for some $v \in \mathbb{R}^2$, $v \neq 0$. If (with some abuse of notation) $v = (1, \kappa)$, $\kappa \in \mathbb{R}$, then relation (2.6) means that $f_1 = \kappa f_0$. If $v = (0, 1)$ then (2.6) means that $f_0 = 0$ which may be identified with the case $\kappa = \infty$. $\square$

Remark. Let $\mathfrak{q}_0$ be the closure of the quadratic form associated to the semibounded symmetric operator $\tilde{H}_0$. Only the self-adjoint extension $\tilde{H}_0(\infty)$ has the property that all functions from its domain have no singularity at the point $\xi = 1$ and belong to the form domain of $\mathfrak{q}_0$. It follows that $\tilde{H}_0(\infty)$ is the Friedrichs extension of $\tilde{H}_0$ (see, for example, Theorem X.23 in [12] or Theorems 5.34 and 5.38 in [9]).

2.3. The Green function

Let us consider the Friedrichs extension of the operator $\tilde{H}$ in $L^2\left((1,\infty) \times S^1, \mathrm{d}\xi\,\mathrm{d}\phi\right)$ which was introduced in (2.4). The resulting self-adjoint operator is in fact the Hamiltonian for the impurity free case. The corresponding Green function $\mathcal{G}_z$ is the generalized kernel of the Hamiltonian, and it should obey the equation

$$(\tilde{H} - z)\mathcal{G}_z(\xi, \phi; \xi', \phi') = \delta(\xi - \xi')\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \delta(\xi - \xi')\mathrm{e}^{im(\phi - \phi')}.$$

If we suppose $\mathcal{G}_z$ to be of the form

$$\mathcal{G}_z(\xi, \phi; \xi', \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \mathcal{G}_z^m(\xi, \xi')\mathrm{e}^{im(\phi - \phi')}, \tag{2.7}$$

then, for all $m \in \mathbb{Z}$,

$$(\tilde{H}_m - z)\mathcal{G}_z^m(\xi, \xi') = \delta(\xi - \xi'). \tag{2.8}$$

Let us consider an arbitrary fixed ξ', and set

$$\mu = m,\ 4\theta = -\frac{a^4\omega^2}{4},\ \lambda = -z - \frac{1}{4}.$$

Then for all $\xi \neq \xi'$ equation (2.8) takes the standard form of the differential equation of spheroidal functions (A.1). As one can see from (A.8), the solution which is square integrable near infinity equals $S_\nu^{|m|(3)}(\xi, -a^4\omega^2/16)$. Furthermore, the solution which is square integrable near $\xi = 1$ equals $Ps_\nu^{|m|}(\xi, -a^4\omega^2/16)$ as one may verify with the aid of the asymptotic formula

$$P_\nu^m(\xi) \sim \frac{\Gamma(\nu + m + 1)}{2^{m/2}\, m!\, \Gamma(\nu - m + 1)} (\xi - 1)^{m/2} \quad \text{as } \xi \to 1+,\ \text{for } m \in \mathbb{N}_0.$$

We conclude that the mth partial Green function equals

$$\mathcal{G}_z^m(\xi,\xi') = -\frac{1}{(\xi^2-1)\mathscr{W}(Ps_\nu^{|m|},S_\nu^{|m|(3)})}\,Ps_\nu^{|m|}\left(\xi_<,-\frac{a^4\omega^2}{16}\right)S_\nu^{|m|(3)}\left(\xi_>,-\frac{a^4\omega^2}{16}\right) \tag{2.9}$$

where the symbol $\mathscr{W}(Ps_\nu^{|m|},S_\nu^{|m|(3)})$ denotes the Wronskian, and $\xi_<$, $\xi_>$ are respectively the smaller and the greater of ξ and ξ'. By the general Sturm-Liouville theory, the factor $(\xi^2-1)\mathscr{W}(Ps_\nu^{|m|},S_\nu^{|m|(3)})$ is constant. Since $\mathcal{G}_z^m = \mathcal{G}_z^{-m}$ decomposition (2.7) may be simplified,

$$\mathcal{G}_z(\xi,\phi;\xi',\phi') = \frac{1}{2\pi}\,\mathcal{G}_z^0(\xi,\xi') + \frac{1}{\pi}\sum_{m=1}^{\infty}\mathcal{G}_z^m(\xi,\xi')\cos\left[m(\phi-\phi')\right]. \tag{2.10}$$

2.4. The Krein Q-function

The Krein Q-function plays a crucial role in the spectral analysis of impurities. It is defined at a point of the configuration space as the regularized Green function evaluated at this point. Here we deal with the impurity located in the center of the dot ($\xi=1$, ϕ arbitrary), and so, by definition,

$$Q(z) := \mathcal{G}_z^{reg}(1,0;1,0).$$

Due to the rotational symmetry,

$$\mathcal{G}_z(\xi) := \mathcal{G}_z(\xi,\phi;1,0) = \mathcal{G}_z(\xi,\phi;1,\phi) = \mathcal{G}_z(\xi,0;1,0) = \frac{1}{2\pi}\,\mathcal{G}_z^0(\xi,1),$$

and hence

$$(\tilde{H}_0 - z)\mathcal{G}_z(\xi) = 0, \quad \text{for } \xi\in(1,\infty).$$

Let us note that from the explicit formula (2.9), one can deduce that the coefficients $\mathcal{G}_z^m(\xi,1)$ in the series in (2.10) vanish for $m=1,2,3,\ldots$. The solution to this equation is

$$\mathcal{G}_z(\xi) \propto S_\nu^{0(3)}\left(\xi,-\frac{a^4\omega^2}{16}\right).$$

The constant of proportionality can be determined with the aid the following theorem which we reproduce from [13].

Theorem 2.3. *Let $d(x,y)$ denote the geodesic distance between points x,y of a two-dimensional manifold X of bounded geometry. Let*

$$U\in\mathcal{P}(X) := \left\{U:\ U_+ := \max(U,0)\in L^{p_0}_{loc}(X),\ U_- := \max(-U,0)\in\sum_{i=1}^{n}L^{p_i}(X)\right\}$$

for an arbitrary $n\in\mathbb{N}$ and $2\le p_i\le\infty$. Then the Green function $\mathcal{G}_U$ of the Schrödinger operator $H_U = -\Delta_{LB}+U$ has the same on-diagonal singularity as that for the Laplace-Beltrami operator itself, i.e.,

$$\mathcal{G}_U(\zeta;x,y) = \frac{1}{2\pi}\log\frac{1}{d(x,y)} + \mathcal{G}_U^{reg}(\zeta;x,y)$$

where $\mathcal{G}_U^{reg}$ is continuous on $X\times X$.

Let us denote by $\mathcal{G}_z^H$ and $Q^H(z)$ the Green function and the Krein Q-function for the Friedrichs extension of H, respectively. Since $\tilde{H} = a^2 H$ and $(\tilde{H} - z)\mathcal{G}_z = \delta$, we have

$$\mathcal{G}_z^H(\xi, \phi; \xi', \phi') = a^2 \mathcal{G}_{a^2 z}(\xi, \phi; \xi', \phi'),\ Q^H(z) = a^2 Q(a^2 z).$$

One may verify that

$$\log d(\varrho, 0; \vec{0}) = \log \varrho = \log(a \operatorname{arg}\cosh \xi) = \frac{1}{2} \log\big(2a^2(\xi - 1)\big) + O(\xi - 1)$$

as $\varrho \to 0+$ or, equivalently, $\xi \to 1+$. Finally, for the divergent part

$$F(\xi, \xi') := \mathcal{G}_z(\xi, \phi; \xi', \phi) - \mathcal{G}_z^{reg}(\xi, \phi; \xi', \phi) = \mathcal{G}_z(\xi, 0; \xi', 0) - \mathcal{G}_z^{reg}(\xi, 0; \xi', 0)$$

of the Green function $\mathcal{G}_z$ we obtain the expression

$$F(\xi, 1) = -\frac{1}{4\pi a^2} \log\big(2a^2(\xi - 1)\big)\,. \tag{2.11}$$

From the above discussion, it follows that the Krein Q-function depends on the coefficients α, β in the asymptotic expansion

$$S_\nu^{0(3)}\left(\xi, -\frac{a^4\omega^2}{16}\right) = \alpha \log(\xi - 1) + \beta + o(1) \quad \text{as } \xi \to 1+, \tag{2.12}$$

and equals

$$Q(z) = -\frac{\beta}{4\pi a^2 \alpha} + \frac{\log(2a^2)}{4\pi a^2}\,. \tag{2.13}$$

To determine α, β we need relation (A.10) for the radial spheroidal function of the third kind. For ν and $\nu + 1/2$ being non-integer, formula (A.12) implies that

$$\begin{aligned} S_\nu^{0(1)}(\xi, \theta) &= \frac{\sin(\nu\pi)}{\pi}\, \mathrm{e}^{-i\pi(\nu+1)} K_\nu^0(\theta) Qs_{-\nu-1}^0(\xi, \theta), \\ S_{-\nu-1}^{0(1)}(\xi, \theta) &= \frac{\sin(\nu\pi)}{\pi}\, \mathrm{e}^{i\pi\nu} K_{-\nu-1}^0(\theta) Qs_\nu^0(\xi, \theta). \end{aligned} \tag{2.14}$$

Applying the symmetry relation (A.5) for expansion coefficients, we derive that

$$\begin{aligned} Qs_{-\nu-1}^0(\xi, \theta) &= \sum_{r=-\infty}^{\infty} (-1)^r a_{-\nu-1,r}^0(\theta) Q_{-\nu-1+2r}^0(\xi) \\ &= \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,r}^0(\theta) Q_{-\nu-1-2r}^0(\xi). \end{aligned}$$

Using the asymptotic formulae (see [5])

$$Q_\nu^0(\xi) = -\frac{1}{2} \log \frac{\xi - 1}{2} + \Psi(1) - \Psi(\nu + 1) + O((\xi - 1)\log(\xi - 1))\,,$$

the series expansion in (A.11) and formulae (2.14), we deduce that, as $\xi \to 1+$,

$$\begin{aligned} S_{\nu}^{0(1)}(\xi,\theta) \sim & -\frac{\sin(\nu\pi)}{\pi}\,\mathrm{e}^{-i\pi(\nu+1)}K_{\nu}^{0}(\theta) \\ & \times\left[s_{\nu}^{0}(\theta)^{-1}\left(\frac{1}{2}\log\frac{\xi-1}{2}-\Psi(1)+\pi\cot(\nu\pi)\right)+\Psi s_{\nu}(\theta)\right], \\ S_{-\nu-1}^{0(1)}(\xi,\theta) \sim & -\frac{\sin(\nu\pi)}{\pi}\,\mathrm{e}^{i\pi\nu}K_{-\nu-1}^{0}(\theta) \\ & \times\left[s_{\nu}^{0}(\theta)^{-1}\left(\frac{1}{2}\log\frac{\xi-1}{2}-\Psi(1)\right)+\Psi s_{\nu}(\theta)\right], \end{aligned}$$

where the coefficients $s_n^{\mu}(\theta)$ are introduced in (A.7),

$$\Psi s_{\nu}(\theta) := \sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,r}^{0}(\theta)\Psi(\nu+1+2r),$$

and where we have made use of the following relation for the digamma function: $\Psi(-z)=\Psi(z+1)+\pi\cot(\pi z)$.

We conclude that

$$S_{\nu}^{0(3)}(\xi,\theta) \sim \alpha\log(\xi-1)+\beta+O\left((\xi-1)\log(\xi-1)\right) \quad \text{as } \xi\to 1+,$$

where

$$\begin{aligned} \alpha &= \frac{i\tan(\nu\pi)}{2\pi s_{\nu}^{0}(\theta)}\left(\mathrm{e}^{i\pi\nu}K_{-\nu-1}^{0}(\theta)-\mathrm{e}^{-i\pi(2\nu+3/2)}K_{\nu}^{0}(\theta)\right), \\ \beta &= \alpha\left(-\log 2-2\Psi(1)+2\Psi s_{\nu}(\theta)s_{\nu}^{0}(\theta)\right)+\mathrm{e}^{-2i\pi\nu}s_{\nu}^{0}(\theta)^{-1}K_{\nu}^{0}(\theta). \end{aligned}$$

The substitution for α, β into (2.13) yields

$$\begin{aligned} Q(z) = & -\frac{1}{4\pi a^2}\left(-\log 2-2\Psi(1)+2\,\Psi s_{\nu}\left(-\frac{a^4\omega^2}{16}\right)s_{\nu}^{0}\left(-\frac{a^4\omega^2}{16}\right)\right) \\ & +\frac{1}{2a^2\tan(\nu\pi)}\left(\mathrm{e}^{i\pi(3\nu+3/2)}\,\frac{K_{-\nu-1}^{0}(-\frac{a^4\omega^2}{16})}{K_{\nu}^{0}(-\frac{a^4\omega^2}{16})}-1\right)^{-1}+\frac{\log(2a^2)}{4\pi a^2} \end{aligned} \tag{2.15}$$

where ν is chosen so that

$$\lambda_{\nu}^{0}\left(-\frac{a^4\omega^2}{16}\right) = -z-\frac{1}{4}\,. \tag{2.16}$$

For $\nu=n$ being an integer, we can immediately use the known asymptotic formulae for spheroidal functions (see Section 16.12 in [5]) which yield

$$\begin{aligned} S_{n}^{0(3)}(\xi,\theta) = & \frac{is_n^0(\theta)}{4\sqrt{\theta}K_n^0(\theta)}\,\log(\xi-1)-\frac{is_n^0(\theta)\log 2}{4\sqrt{\theta}K_n^0(\theta)} \\ & +\frac{is_n^0(\theta)^2}{2\sqrt{\theta}K_n^0(\theta)}\sum_{2r\geq -n}(-1)^r a_{n,r}^{0}(\theta)h_{n+2r}+\frac{K_n^0(\theta)}{s_n^0(\theta)}+O(\xi-1), \end{aligned}$$

as $\xi\to 1+$. Here, $h_0=1, h_k=1/1+1/2+\ldots+1/k$. By (2.13), one can calculate the Q-function in this case, too.

2.5. The spectrum of a quantum dot with impurity

The Green function of the Hamiltonian describing a quantum dot with impurity is given by the Krein resolvent formula

$$\mathcal{G}_z^{H(\chi)}(\xi,\phi;\xi',\phi') = \mathcal{G}_z^{H}(\xi,\phi;\xi',\phi') - \frac{1}{Q^H(z)-\chi}\,\mathcal{G}_z^{H}(\xi,0;1,0)\mathcal{G}_z^{H}(1,0;\xi',0)$$

(recall that, due to the rotational symmetry, $\mathcal{G}_z^{H}(\xi,\phi;1,0) = \mathcal{G}_z^{H}(\xi,0;1,0)$). The parameter $\chi := a^2\kappa \in (-\infty,\infty\,]$ determines the corresponding self-adjoint extension $H(\chi)$ of H. In the physical interpretation, this parameter is related to the strength of the δ interaction. Recall that the value $\chi = \infty$ corresponds to the Friedrichs extension of H representing the case with no impurity. This fact is also apparent from the Krein resolvent formula.

The unperturbed Hamiltonian $H(\infty)$ describes a harmonic oscillator on the Lobachevsky plane. As is well known (see, for example, [14]), for the confinement potential tends to infinity as $\varrho \to \infty$, the resolvent of $H(\infty)$ is compact, and the spectrum of $H(\infty)$ is discrete and semibounded. The eigenvalues of $H(\infty)$ are solutions of a scalar equation whose introduction also relies heavily on the theory of spheroidal functions. We are sceptic about the possibility of deriving an explicit formula for the eigenvalues. But the equation turned out to be convenient enough to allow for numerical solutions. A more detailed discussion jointly with a basic numerical analysis is provided in a separate paper [15].

A similar observation about the basic spectral properties (discreteness and semiboundedness) is also true for the operators $H(\chi)$ for any $\chi \in \mathbb{R}$ since, by the Krein resolvent formula, the resolvents for $H(\chi)$ and $H(\infty)$ differ by a rank one operator. Moreover, the multiplicities of eigenvalues of $H(\chi)$ and $H(\infty)$ may differ at most by ± 1 (see [9, Section 8.3]).

A more detailed and rather general analysis which is given in [1] can be carried over to our case almost literally. Denote by σ the set of poles of the function $Q^H(z)$ depending on the spectral parameter z. Note that σ is a subset of $\operatorname{spec}(H(\infty))$. Consider the equation

$$Q^H(z) = \chi. \tag{2.17}$$

Theorem 2.4. *The spectrum of $H(\chi)$ is discrete and consists of four nonintersecting parts S_1, S_2, S_3, S_4 described as follows:*

1. *S_1 is the set of all solutions to equation (2.17) which do not belong to the spectrum of $H(\infty)$. The multiplicity of all these eigenvalues in the spectrum of $H(\chi)$ equals 1.*
2. *S_2 is the set of all $\lambda \in \sigma$ that are multiple eigenvalues of $H(\infty)$. If the multiplicity of such an eigenvalue λ in $\operatorname{spec}(H(\infty))$ equals k then its multiplicity in the spectrum of $H(\chi)$ equals $k-1$.*
3. *S_3 consists of all $\lambda \in \operatorname{spec}(H(\infty))\backslash\sigma$ that are not solutions to equation (2.17). the multiplicities of such an eigenvalue λ in $\operatorname{spec}(H(\infty))$ and $\operatorname{spec}(H(\chi))$ are equal.*

4. *S_4 consists of all $\lambda \in \mathrm{spec}(H(\infty)) \setminus \sigma$ that are solutions to equation (2.17). If the multiplicity of such an eigenvalue λ in $\mathrm{spec}(H(\infty))$ equals k then its multiplicity in the spectrum of $H(\chi)$ equals $k+1$.*

Hence the eigenvalues of $H(\chi)$, $\chi \in \mathbb{R}$, different from those of the unperturbed Hamiltonian $H(\infty)$ are solutions to (2.17). As far as we see it, this equation can be solved only numerically. We have postponed a systematic numerical analysis of equation (2.17) to a subsequent work. Note that the Krein Q-function (2.15) is in fact a function of ν, and hence dependence (2.16) of the spectral parameter z on ν is fundamental. In this context, it is quite useful to know for which values of ν the spectral parameter z is real. A partial answer is given by Proposition A.1.

3. Conclusion

We have proposed a Hamiltonian describing a quantum dot in the Lobachevsky plane to which we added an impurity modeled by a δ potential. Formulas for the corresponding Q- and Green functions have been derived. Further analysis of the energy spectrum may be accomplished for some concrete values of the involved parameters (by which we mean the curvature a and the oscillator frequency ω) with the aid of numerical methods.

Appendix: Spheroidal functions

Here we follow the source [5]. Spheroidal functions are solutions to the equation

$$(1-\xi^2)\frac{\partial^2\psi}{\partial\xi^2} - 2\xi\frac{\partial\psi}{\partial\xi} + \left[\lambda + 4\theta(1-\xi^2) - \mu^2(1-\xi^2)^{-1}\right]\psi = 0, \tag{A.1}$$

where all parameters are in general complex numbers. There are two solutions that behave like ξ^ν times a single-valued function and $\xi^{-\nu-1}$ times a single-valued function at ∞. The exponent ν is a function of λ, θ, μ, and is called the characteristic exponent. Usually, it is more convenient to regard λ as a function of ν, μ and θ. We shall write $\lambda = \lambda_\nu^\mu(\theta)$. If ν or μ is an integer we denote it by n or m, respectively. The functions $\lambda_\nu^\mu(\theta)$ obey the symmetry relations

$$\lambda_\nu^\mu(\theta) = \lambda_\nu^{-\mu}(\theta) = \lambda_{-\nu-1}^\mu(\theta) = \lambda_{-\nu-1}^{-\mu}(\theta). \tag{A.2}$$

A first group of solutions (radial spheroidal functions) is obtained as expansions in series of Bessel functions,

$$S_\nu^{\mu(j)}(\xi,\theta) = (1-\xi^{-2})^{-\mu/2} s_\nu^\mu(\theta) \sum_{r=-\infty}^{\infty} a_{\nu,r}^\mu \psi_{\nu+2r}^{(j)}(2\theta^{1/2}\xi), \tag{A.3}$$

$j = 1, 2, 3, 4$, where the factors $s_\nu^\mu(\theta)$ are determined below and

$$\psi_\nu^{(1)}(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\, J_{\nu+1/2}(\zeta), \quad \psi_\nu^{(2)}(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\, Y_{\nu+1/2}(\zeta),$$
$$\psi_\nu^{(3)}(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\, H_{\nu+1/2}^{(1)}(\zeta), \quad \psi_\nu^{(4)}(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\, H_{\nu+1/2}^{(2)}(\zeta).$$

The coefficients $a_{\nu,r}^\mu(\theta)$ (denoted only a_r for the sake of simplicity) satisfy a three term recurrence relation

$$\begin{aligned}&\frac{(\nu+2r-\mu)(\nu+2r-\mu-1)}{(\nu+2r-3/2)(\nu+2r-1/2)}\,\theta a_{r-1} + \frac{(\nu+2r+\mu+2)(\nu+2r+\mu+1)}{(\nu+2r+3/2)(\nu+2r+5/2)}\,\theta a_{r+1}\\ &+ \left[\lambda_\nu^\mu(\theta) - (\nu+2r)(\nu+2r+1) + \frac{(\nu+2r)(\nu+2r+1)+\mu^2-1}{(\nu+2r-1/2)(\nu+2r+3/2)}\,2\theta\right] a_r = 0.\end{aligned} \tag{A.4}$$

Here and in what follows we assume that $\nu+1/2$ is not an integer (to our knowledge, the omitted case is not yet fully investigated).

The coefficients $a_{\nu,r}^\mu(\theta)$ may be chosen such that

$$a_{\nu,0}^\mu(\theta) = a_{-\nu-1,0}^\mu(\theta) = a_{\nu,0}^{-\mu}(\theta),$$

and so (see (A.2))

$$a_{\nu,r}^\mu(\theta) = a_{-\nu-1,-r}^\mu(\theta) = \frac{(\nu-\mu+1)_{2r}}{(\nu+\mu+1)_{2r}}\, a_{\nu,r}^{-\mu}(\theta) \tag{A.5}$$

where $(a)_r := a(a+1)(a+2)\ldots(a+r-1) = \Gamma(a+r)/\Gamma(a)$, $(a)_0 := 1$. Equation (A.4) leads to a convergent infinite continued fraction and this way one can prove that

$$\lim_{r\to\infty} \frac{r^2 a_r}{a_{r-1}} = \lim_{r\to-\infty} \frac{r^2 a_r}{a_{r+1}} = \frac{\theta}{4}. \tag{A.6}$$

From (A.6) and the asymptotic formulae for Bessel functions, it follows that (A.3) converges if $|\xi| > 1$.

If we set in (A.3)

$$s_\nu^\mu(\theta) = \left[\sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,r}^\mu(\theta)\right]^{-1} \tag{A.7}$$

then

$$S_\nu^{\mu(j)}(\xi,\theta) \sim \psi_\nu^{(j)}(2\theta^{1/2}\xi), \quad \text{for } |\arg(\theta^{1/2}\xi)| < \pi \quad \text{as } \xi \to \infty.$$

We have the asymptotic forms, valid as $\xi \to \infty$,

$$\begin{aligned}&S_\nu^{\mu(3)}(\xi,\theta) = \frac{1}{2}\,\theta^{-1/2}\xi^{-1}\mathrm{e}^{i(2\theta^{1/2}\xi-\nu\pi/2-\pi/2)}[1+O(|\xi|^{-1})],\\ &\text{for } -\pi < \arg(\theta^{1/2}\xi) < 2\pi,\end{aligned} \tag{A.8}$$

and

$$S_\nu^{\mu(4)}(\xi,\theta) = \frac{1}{2}\,\theta^{-1/2}\xi^{-1}\mathrm{e}^{-i(2\theta^{1/2}\xi-\nu\pi/2-\pi/2)}[1+O(|\xi|^{-1})],\qquad \text{for } -2\pi < \arg(\theta^{1/2}\xi) < \pi. \tag{A.9}$$

The radial spheroidal functions satisfy the relation

$$S_\nu^{\mu(3)} = \frac{1}{i\cos(\nu\pi)}\left(S_{-\nu-1}^{\mu(1)} + i\,\mathrm{e}^{-i\pi\nu}S_\nu^{\mu(1)}\right). \tag{A.10}$$

The radial spheroidal functions are especially useful for large ξ; the larger is the ξ the better is the convergence of the expansion. To obtain solutions useful near ±1, and even on the segment $(-1,1)$, one uses expansions in series in Legendre functions,

$$\begin{aligned} Ps_\nu^\mu(\xi,\theta) &= \sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,r}^\mu(\theta)P_{\nu+2r}^\mu(\xi),\\ Qs_\nu^\mu(\xi,\theta) &= \sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,r}^\mu(\theta)Q_{\nu+2r}^\mu(\xi). \end{aligned} \tag{A.11}$$

These solutions are called the angular spheroidal functions and are related to the radial spheroidal functions by the following formulae:

$$\begin{aligned} S_\nu^{\mu(1)}(\xi,\theta) &= \pi^{-1}\sin[(\nu-\mu)\pi]\mathrm{e}^{-i\pi(\nu+\mu+1)}K_\nu^\mu(\theta)Qs_{-\nu-1}^\mu(\xi,\theta),\\ S_n^{m(1)}(\xi,\theta) &= K_n^m(\theta)Ps_n^m(\xi,\theta), \end{aligned} \tag{A.12}$$

where $K_\nu^\mu(\theta)$ can be expressed as a series in coefficients $a_{\nu,r}^\mu(\theta)$, and sometimes it is called the joining factor. In more detail, for any $k\in\mathbb{Z}$ it holds true that

$$K_\nu^\mu(\theta) = \frac{1}{2}\left(\frac{\theta}{4}\right)^{\nu/2+k}\Gamma(1+\nu-\mu+2k)\,\mathrm{e}^{(\nu+k)\pi i}s_\nu^\mu(\theta) \times \frac{\displaystyle\sum_{r=-\infty}^{k}\frac{(-1)^r a_{\nu,r}^\mu(\theta)}{(k-r)!\,\Gamma(\nu+k+r+3/2)}}{\displaystyle\sum_{r=k}^{\infty}\frac{(-1)^r a_{\nu,r}^\mu(\theta)}{(r-k)!\,\Gamma(1/2-\nu-k-r)}}.$$

Proposition A.1. *Let $\nu,\theta\in\mathbb{R}$ and set $\mu=0$. Then $\lambda_\nu^0(\theta)\in\mathbb{R}$.*

Proof. To simplify the notation we denote, in (A.4),

$$\begin{aligned} \alpha_r^{\mu,\nu}(\theta) &= \frac{(\nu+2r+\mu+2)(\nu+2r+\mu+1)}{(\nu+2r+3/2)(\nu+2r+5/2)}\,\theta,\\ \beta_r^{\mu,\nu}(\theta) &= -(\nu+2r)(\nu+2r+1) + \frac{(\nu+2r)(\nu+2r+1)+\mu^2-1}{(\nu+2r-1/2)(\nu+2r+3/2)}\,2\theta,\\ \gamma_r^{\mu,\nu}(\theta) &= \frac{(\nu+2r-\mu)(\nu+2r-\mu-1)}{(\nu+2r-3/2)(\nu+2r-1/2)}\,\theta. \end{aligned}$$

The resulting formula may be written in the matrix form,

$$\begin{pmatrix} \ddots & & & & & \\ & \gamma_{-1} & \beta_{-1} & \alpha_{-1} & & \\ & & \gamma_0 & \beta_0 & \alpha_0 & \\ & & & \gamma_1 & \beta_1 & \alpha_1 \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{pmatrix} = -\lambda \begin{pmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{pmatrix} \qquad \text{(A.13)}$$

where we have omitted the fixed indices.

As one can see,

$$\gamma^{0,\nu}_{r+1}(\theta) = \frac{\nu + 2r + 5/2}{\nu + 2r + 1/2}\,\alpha^{0,\nu}_r(\theta)$$

and so

$$\frac{\nu + 2r + 1/2}{\nu + 2r - 3/2}\alpha^{0,\nu}_{r-1}(\theta)a^0_{\nu,r-1}(\theta) + \beta^{0,\nu}_r(\theta)a^0_{\nu,r}(\theta) + \alpha^{0,\nu}_r(\theta)a^0_{\nu,r+1}(\theta) = -\lambda^0_\nu(\theta)a^0_{\nu,r}(\theta).$$

Substitution $a^0_{\nu,r} = L_r(\nu)\tilde{a}^0_{\nu,r}$, where $L_r(\nu)$ are non-zero constants, yields

$$\frac{\nu + 2r + 1/2}{\nu + 2r - 3/2}\alpha^{0,\nu}_{r-1}(\theta)\tilde{a}^0_{\nu,r-1}(\theta)\frac{L_{r-1}(\nu)}{L_r(\nu)} + \beta^{0,\nu}_r(\theta)\tilde{a}^0_{\nu,r}(\theta) + \alpha^{0,\nu}_r(\theta)\tilde{a}^0_{\nu,r+1}(\theta)\frac{L_{r+1}(\nu)}{L_r(\nu)}$$
$$= -\lambda^0_\nu(\theta)\tilde{a}^0_{\nu,r}(\theta).$$

We require the matrix in (A.13) to be symmetric in the new coordinates $\{\tilde{a}_r\}$. This implies that

$$\frac{\nu + 2r + 1/2}{\nu + 2r - 3/2}\frac{L_{r-1}(\nu)}{L_r(\nu)} = \frac{L_r(\nu)}{L_{r-1}(\nu)}.$$

For $r \notin (-\nu/2 - 1/4, -\nu/2 + 3/4)$, the solution is $L_r(\nu) = \sqrt{|\nu + 2r + 1/2|}$. For $r_0 \equiv r \in (-\nu/2 - 1/4, -\nu/2 + 3/4)$, there is no real solution and so we set $L_{r_0}(\nu) = \sqrt{|\nu + 2r_0 + 1/2|}$ and make another transformation of coordinates:

$$\tilde{\tilde{a}}_r = \begin{cases} -\tilde{a}_r & \text{for } r = r_0 - (2k-1),\ k \in \mathbb{N}, \\ \tilde{a}_r & \text{for all other } r. \end{cases}$$

Relation (A.13) can be viewed as an eigenvalue equation with a symmetric matrix in the coordinate system $\{\tilde{\tilde{a}}_k\}$, hence $\lambda^0_\nu(\theta)$ must be real. □

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V. Geyler
Department of Mathematics
Mordovian State University
Saransk, Russia

P. Šťovíček
Department of Mathematics
Faculty of Nuclear Sciences
Czech Technical University
Prague, Czech Republic
e-mail: `stovicek@kmalpha.fjfi.cvut.cz`

M. Tušek
Department of Mathematics
Faculty of Nuclear Sciences
Czech Technical University
Prague, Czech Republic
e-mail: tusekmat@km1.fjfi.cvut.cz